Problems from section 2.2.

#3. Prove Theorem 2.12iv: Suppose that $\{x_n\}$ and $\{y_n\}$ are real sequences such that $y_n \neq 0$ and $\lim_{n\to\infty} y_n \neq 0$. If $\{x_n\}$ and $\{y_n\}$ converge, then $\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{\lim_{n\to\infty} x_n}{\lim_{n\to\infty} y_n}$.

Proof. Let $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. It is sufficient to show that $\frac{x_n}{y_n} - \frac{x}{y} \to 0$. To this end, observe that

$$\frac{x_n}{y_n} - \frac{x}{y} = \frac{x_n y - x y_n}{y y_n} = \frac{y_n(x_n - x) - x_n(y_n - y)}{y y_n} = [y_n(x_n - x) - x_n(y_n - y)]\,\frac{1}{y y_n}.$$

This last part is motivated by the method given in the book for the proof of Theorem 2.12iii. If we can argue that $y_n(x_n - x) - x_n(y_n - y) \to 0$ and $\frac{1}{y y_n}$ is bounded, then part (ii) of the Squeeze Theorem would imply that $\frac{x_n}{y_n} - \frac{x}{y} \to 0$ as well, which implies that $\frac{x_n}{y_n} \to \frac{x}{y}$.

$y_n(x_n - x) - x_n(y_n - y) \to 0$: To see this, we show that both $x_n(y_n - y) \to 0$ and $y_n(x_n - x) \to 0$ and use part(i) of Theorem 2.12. To see that $\lim x_n(y_n - y) = 0$, we apply part (ii) of the Squeeze Law: observe that $\{x_n\}$ being convergent implies $\{x_n\}$ is bounded (by Theorem 2.8) and $y_n - y \to 0$ since $y_n \to y$. A similar argument shows $y_n(x_n - x) \to 0$.

$\left\{\frac{1}{y y_n}\right\}$ is bounded: Since $y \neq 0$, let ϵ be any real number satisfying $0 < \epsilon < |y|$. Since $y_n \to y$, $\exists\ N \in \mathbb{N}$ s.t. $\forall$ $n \geq N$, $|y_n - y| < \epsilon$. That is, $-\epsilon < y_n - y < \epsilon$. That is, $y - \epsilon < y_n < y + \epsilon$. Since $0 < \epsilon < |y|$, you can easily convince yourself that $y - \epsilon$ and $y + \epsilon$ have the same sign (consider the cases $y < 0$ and $y > 0$). Further, because both $y - \epsilon$ and $y + \epsilon$ have the same sign and because $y - \epsilon < y_n < y + \epsilon$, we can conclude that $\forall\ n \geq N$, $\frac{1}{y+\epsilon} < \frac{1}{y_n} < \frac{1}{y-\epsilon}$. This establishes that $\left\{\frac{1}{y_n}\right\}$ is bounded. Further, since $y \neq 0$, then $\frac{1}{y}$ is just some finite value, and so $\left\{\frac{1}{y y_n}\right\}$ is bounded as well.

Thus, we have established that $y_n(x_n - x) - x_n(y_n - y) \to 0$ and $\left\{\frac{1}{y y_n}\right\}$ is bounded. Further, we pointed out at the beginning that $\frac{x_n}{y_n} - \frac{x}{y} = [y_n(x_n - x) - x_n(y_n - y)]\,\frac{1}{y y_n}$. Therefore, by part (ii) of the Squeeze Law, $\frac{x_n}{y_n} - \frac{x}{y} \to 0$, which implies that $\frac{x_n}{y_n} \to \frac{x}{y}$. $\square$

#4. Suppose $x \in \mathbb{R}$, $x_n \geq 0$, and $x_n \to x$ as $n \to \infty$. Prove that $\sqrt{x_n} \to \sqrt{x}$ as $n \to \infty$.

Proof. First let us consider the case that $x > 0$. It suffices to show that $\sqrt{x_n} - \sqrt{x} \to 0$. To this end, observe that

$$\sqrt{x_n} - \sqrt{x} = \left(\sqrt{x_n} - \sqrt{x}\right) \frac{\sqrt{x_n} + \sqrt{x}}{\sqrt{x_n} + \sqrt{x}} = (x_n - x) \cdot \frac{1}{\sqrt{x_n} + \sqrt{x}}.$$

In a similar fashion as the last problem, we will employ part (ii) of the Squeeze Law: First, since $x_n \to x$, we know $x_n - x \to 0$. Also, note that

$$0 < \frac{1}{\sqrt{x_n} + \sqrt{x}} < \frac{1}{\sqrt{x}}$$

because the fraction on the right has smaller positive denominator. Since 0 and $\frac{1}{\sqrt{x}}$ are constants, we have bounds on the sequence terms $\dfrac{1}{\sqrt{x_n} + \sqrt{x}}$. Therefore, by part (ii) of the Squeeze Law,

$$\sqrt{x_n} - \sqrt{x} \to 0.$$

In the case that $x = 0$, let use use the fact that $x_n \to 0$ to observe that for $\epsilon > 0$, $\exists\ N \in \mathbb{N}$ s.t. $\forall\ n \geq N$, $|x_n - 0| < \epsilon^2$. Since $x_n \geq 0$, this last inequalities says $x_n < \epsilon^2$. Furthermore, since $x_n \geq 0$ by hypothesis, we now have that $\forall\ n \geq N$,

$$0 \leq x_n < \epsilon^2.$$

Applying (8) from page 7 of the text, we see that $\forall\ n \geq N$,

$$0 \leq \sqrt{x_n} < \sqrt{\epsilon^2}.$$

That is, $\forall\ n \geq N$,

$$0 \leq \sqrt{x_n} < \epsilon.$$

That is, $\forall\ n \geq N$,

$$|\sqrt{x_n} - 0| < \epsilon.$$

That is, $\sqrt{x_n} \to 0$. □